Two methods for computing $f*g$:

(1) $f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$ (for computers)

(2) $f*g = \mathcal{L}^{-1}\{\mathcal{L}\{f\}\cdot\mathcal{L}\{g\}\}$ (for people)

Important Property: $(f*g)(0) = 0$ ALWAYS

(Convolutions are always 0 at $t=0$)

⟹ Easy way to check for mistakes.

EX: Compute $1*1$.

$$1*1 = \mathcal{L}^{-1}\{\mathcal{L}\{1\}\cdot\mathcal{L}\{1\}\} = \mathcal{L}^{-1}\{1/s\cdot 1/s\}$$
$$= \mathcal{L}^{-1}\{1/s^2\} = \boxed{t}$$

EX: Compute $1*t$.

$$1*t = \mathcal{L}^{-1}\{\mathcal{L}\{1\}\cdot\mathcal{L}\{t\}\} = \mathcal{L}^{-1}\{1/s\cdot 1/s^2\}$$
$$= \mathcal{L}^{-1}\{1/s^3\} = \boxed{1/2\, t^2}$$

→ More generally: $1*t^n = \frac{1}{n+1}\, t^{n+1}$

EX: Compute $1*\sin t$

$$1*\sin t = \mathcal{L}^{-1}\{\mathcal{L}\{1\}\cdot\mathcal{L}\{\sin t\}\} = \mathcal{L}^{-1}\left\{1/s\cdot\frac{1}{s^2+1}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{-1\cdot s}{s^2+1}\right\}$$
$$= 1 - \cos t$$
$$\hookrightarrow \boxed{-\cos t + 1}$$

$A(s^2+1) + (Bs+C)s = 1$

$A=1$ $A+B=0$

$C=0$ $B=-1$

EX: Compute $1*\cos t$

$$1*\cos t = \mathcal{L}^{-1}\{\mathcal{L}\{1\}\cdot\mathcal{L}\{\cos t\}\} = \mathcal{L}^{-1}\left\{1/s\cdot\frac{s}{s^2+1}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$= \boxed{\sin t}$$

General Rule: $1*f(t) = \int_0^t f(\tau)\, d\tau$

= anti-derivative of $f(t)$ with C chosen so that value at 0 is 0.

EX: Compute $1*e^{4t}$

$$\left.\begin{aligned}\int e^{4t}\,dt &= 1/4\, e^{4t} + C\\ 0 = 1/4\, e^0 + C &\Rightarrow C = -1/4\end{aligned}\right\}\ \boxed{1/4\, e^{4t} - 1/4}$$

EX: Compute $t * t$

$$t * t = \mathcal{L}^{-1}\{\mathcal{L}\{t\}\cdot\mathcal{L}\{t\}\} = \mathcal{L}^{-1}\{1/s^2 \cdot 1/s^2\}$$
$$= \mathcal{L}^{-1}\{\tfrac{1}{s^4}\} = \boxed{1/3!\ t^3}$$

Note: $t * f(t) = 1 * 1 * f(t)$
$= 2\times$ anti-derivative of $f(t)$

EX Compute $t * \sin t$

$$t * \sin t = 1 * 1 * \sin t$$
$$= 1 * (-\cos t + 1)$$
$$= \boxed{-\sin t + t + 0}$$

— alternate computation —

$$t * \sin t = \mathcal{L}^{-1}\{\mathcal{L}\{t\}\cdot\mathcal{L}\{\sin t\}\}$$
$$= \mathcal{L}^{-1}\{1/s^2 \cdot \tfrac{1}{s^2+1}\}$$
$$= \mathcal{L}^{-1}\{\tfrac{A}{s^2} + \tfrac{B}{s} + \tfrac{Cs+D}{s^2+1}\}$$
$$= \mathcal{L}^{-1}\{\tfrac{1}{s^2} + \tfrac{0}{s} + \tfrac{0s+(-1)}{s^2+1}\}$$
$$= \boxed{t - \sin t}$$

$A(s^2+1)$
$+ B(s^2+1)s$
$+ (Cs+D)s^2 = 1$

$A = 1$ $A+D=0$
$B = 0$ $B+C=0$

$D=-1$
$C=0$

EX: Compute $t * e^{5t}$

$$t * e^{5t} = 1 * 1 * e^{5t}$$
$$= 1 * (1/5\, e^{5t} - 1/5)$$
$$= \boxed{1/25\, e^{5t} - 1/5\, t - 1/25}$$

— alternate computation —

$$t * e^{5t} = \mathcal{L}^{-1}\{\mathcal{L}\{t\}\cdot\mathcal{L}\{e^{5t}\}\}$$
$$= \mathcal{L}^{-1}\{1/s^2 \cdot \tfrac{1}{s-5}\}$$
$$= \mathcal{L}^{-1}\{\tfrac{A}{s^2} + \tfrac{B}{s} + \tfrac{C}{s-5}\}$$
$$= \mathcal{L}^{-1}\{-1/5 \cdot 1/s^2 + 1/25\, 1/s + (-1/25)\, 1/(s-5)\}$$
$$= \boxed{-1/5\, t + 1/25 - 1/25\, e^{5t}}$$

$A(s-5)$
$+ B(s-5)s$
$+ Cs^2 = 1$

$-5A = 1$
$A - 5B = 0$
$B + C = 0$

Note: $t^2 * f(t) = 2(1 * 1 * 1 * f(t))$
$= 2$ ($3\times$ anti-derivative of $f(t)$)

... More generally
$$t^n * f(t) = n * (n-1) * \cdots * 2 * 1 * 1 * f(t)$$
$= (n!)$ ($(n+1)\times$ anti-deriv of f)

EX: $t^2 * \cos t = 2 * 1 * 1 * \cos t$
$$= 2 * 1 * (\sin t)$$
$$= 2 * (-\cos t + 1)$$
$$= 2(-\sin t + t)$$

Note: $*$ is commutative i.e. $f*g = g*f$

EX: Compute $\cos 2t * t$

$$\begin{aligned}
\cos 2t * t &= t * \cos 2t \\
&= 1 * 1 * \cos 2t \\
&= 1 * \left(\tfrac{1}{2}\sin 2t + 0 \right) \\
&= \boxed{-\tfrac{1}{4}\cos 2t + \tfrac{1}{4}}
\end{aligned}$$

What about other functions?

EX: Compute $\sin t * \cos t$

$$\begin{aligned}
\sin t * \cos t &= \mathcal{L}^{-1}\{\mathcal{L}\{\sin t\}\,\mathcal{L}\{\cos t\}\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \\
&= t\cdot\mathcal{L}^{-1}\left\{-\int\frac{s}{(s^2+1)^2}\,ds\right\} \\
&= t\cdot\mathcal{L}^{-1}\left\{\frac{1}{2}\,\frac{1}{s^2+1}\right\} \\
&= \boxed{t\cdot\tfrac{1}{2}\sin t}
\end{aligned}$$

... This one is pretty tricky...

(Convolutions quickly become impossible to compute.)

EX: Compute $e^{5t} * e^{2t}$

$$\begin{aligned}
e^{5t} * e^{2t} &= \mathcal{L}^{-1}\{\mathcal{L}\{e^{5t}\}\cdot\mathcal{L}\{e^{2t}\}\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s-5}\cdot\frac{1}{s-2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A}{s-5}+\frac{B}{s-2}\right\} \\
&= \mathcal{L}^{-1}\left\{\tfrac{1}{3}\frac{1}{s-5}+\left(-\tfrac{1}{3}\right)\frac{1}{s-2}\right\} \\
&= \boxed{\tfrac{1}{3}e^{5t} - \tfrac{1}{3}e^{2t}}
\end{aligned}$$

$A(s-2) + B(s-5) = 1$

$s=5$: $A(3)=1$

$s=2$: $B(-3)=1$

EX: Compute $e^{2t} * \sin t$

$$\begin{aligned}
e^{2t} * \sin t &= \mathcal{L}^{-1}\{\mathcal{L}\{e^{2t}\}\cdot\mathcal{L}\{\sin t\}\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A}{s-2}+\frac{Bs+C}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\tfrac{1}{5}\frac{1}{s-2}+\left(-\tfrac{1}{5}\right)\frac{s}{s^2+1}+\left(-\tfrac{2}{5}\right)\frac{1}{s^2+1}\right\} \\
&= \boxed{\tfrac{1}{5}e^{2t} - \tfrac{1}{5}\cos t - \tfrac{2}{5}\sin t}
\end{aligned}$$

$A(s^2+1) + (Bs+C)(s-2) = 1$

$s=2$: $A(5)=1 \rightsquigarrow A=\tfrac{1}{5}$

(s^2) $A+B=0 \rightsquigarrow B=-\tfrac{1}{5}$

(s) $-2B+C=0 \rightsquigarrow C=-\tfrac{2}{5}$

EX: Compute $e^{-3t} * \cos t$

$$\begin{aligned}
e^{-3t} * \cos t &= \mathcal{L}^{-1}\{\mathcal{L}\{e^{-3t}\}\,\mathcal{L}\{\cos t\}\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s+3}\cdot\frac{s}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A}{s+3}+\frac{Bs+C}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\left(-\tfrac{3}{10}\right)\frac{1}{s+3}+\left(\tfrac{3}{10}\right)\frac{s}{s^2+1}+\left(\tfrac{1}{10}\right)\frac{1}{s^2+1}\right\} \\
&= \boxed{-\tfrac{3}{10}e^{-3t} + \tfrac{3}{10}\cos t + \tfrac{1}{10}\sin t}
\end{aligned}$$

$A(s^2+1) + (Bs+C)(s+3) = s$

$s=-3$: $A(10) = -3 \rightsquigarrow A=-\tfrac{3}{10}$

(s^2) $A+B=0 \rightsquigarrow B=\tfrac{3}{10}$

(s) $3B+C=1 \rightsquigarrow C=\tfrac{1}{10}$

Convolutions with Step & Impulse functions:

EX: Compute $1 * \delta_2(t)$

$$1 * \delta_2(t) = \mathcal{L}^{-1}\{\mathcal{L}\{1\} \cdot \mathcal{L}\{\delta_2\}\}$$
$$= \mathcal{L}^{-1}\{ 1/s \cdot e^{-2s} \}$$
$$= u_2(t)$$

EX Compute $t^2 * \delta_5(t)$

$$t^2 * \delta_5(t) = \mathcal{L}^{-1}\{\mathcal{L}\{t^2\} \cdot \mathcal{L}\{\delta_5\}\}$$
$$= \mathcal{L}^{-1}\{ 2/s^3 \cdot e^{-5s} \}$$
$$= u_5(t) \cdot (t-5)^2$$

EX Compute $\cos t * \delta_3(t)$

$$\cos t * \delta_3(t) = \mathcal{L}^{-1}\{\mathcal{L}\{\cos t\} \mathcal{L}\{\delta_3\}\}$$
$$= \mathcal{L}^{-1}\{ \tfrac{s}{s^2+1} \cdot e^{-3s} \}$$
$$= u_3(t) \cdot \cos(t-3)$$

EX Compute $u_2(t) * \delta_5(t)$

$$u_2(t) * \delta_5(t) = \mathcal{L}^{-1}\{ e^{-2s}\, 1/s \cdot e^{-5s} \}$$
$$= u_7(t)$$

EX: Compute $\delta_3(t) * \delta_5(t)$

$$\delta_3(t) * \delta_5(t) = \mathcal{L}^{-1}\{\mathcal{L}\{\delta_3\} \cdot \mathcal{L}\{\delta_5\}\}$$
$$= \mathcal{L}^{-1}\{ e^{-3s} \cdot e^{-5s} \} = \mathcal{L}^{-1}\{ e^{-8s} \}$$
$$= \delta_8(t)$$

EX: Compute $1 * u_2(t)$

$$1 * u_2(t) = \mathcal{L}^{-1}\{\mathcal{L}\{1\} \cdot \mathcal{L}\{u_2\}\}$$
$$= \mathcal{L}^{-1}\{ 1/s \cdot e^{-2s}\, 1/s \} = \mathcal{L}^{-1}\{ e^{-2s} \tfrac{1}{s^2} \}$$
$$= u_2(t) \cdot (t-2)$$

EX Compute $\cos t * u_5(t)$

$$\cos t * u_5(t) = \mathcal{L}^{-1}\{\mathcal{L}\{\cos t\} \mathcal{L}\{u_5\}\}$$
$$= \mathcal{L}^{-1}\{ \tfrac{s}{s^2+1} \cdot e^{-5s}\, 1/s \}$$
$$= u_5(t) \cdot \sin(t-3)$$

$$\rightsquigarrow f(t) * u_c(t) = u_c(t) \cdot \left(\begin{array}{c} \text{shifted} \\ \text{anti-deriv.} \end{array} \right)$$

EX: Compute $u_3(t) * u_5(t)$

$$u_3(t) * u_5(t) = \mathcal{L}^{-1}\{\mathcal{L}\{u_3\} \mathcal{L}\{u_5\}\}$$
$$= \mathcal{L}^{-1}\{ e^{-3s}\, 1/s \cdot e^{-5s}\, 1/s \}$$
$$= u_8(t) \cdot (t-8)$$